TEMPERATURE REGIME OF A THIN PLATE HEATED BY AN IMPULSIVE LOCALIZED ENERGY SOURCE

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An examination is made of the problem of determining the temperature field of a thin plate, in a finite circular region of which a short-duration impulsive energy source acts; equations for determining the temperature field and the heat flux are given.

A plate whose thickness δ is much less than its length or width receives heat flux $q(\tau)$ per unit area on a circular region of radius R from an extraneous energy source. The thermophysical properties of the plate, thermal conductivity λ, thermal diffusivity a, and volume specific heat c_V, are assumed to be known, and independent of temperature. The plate is located in a medium whose temperature t_m is assigned and taken as a reference in the calculation. Because of the small thickness of the plate, the temperature gradient through the plate during the heating may be neglected, and it may be considered that the temperature distribution depends on two parameters, the coordinate r and the time τ. Since absorption of heat from the external source occurs only in the section of the circle of area πR^2, it is expedient to divide the plate into two regions, within which the temperature distribution will be described by functions $u_1(r, \tau)$ and $u_2(r, \tau)$ respectively.

Heat transfer between the plate and the medium occurs according to Newton's law, i.e., the heat flux from the surface of the body to the medium is proportional to the temperature difference between the surface and the medium.

It may be shown that, under the assumptions made, the processes of heat transfer in regions 1 and 2 of the plate obey the equations

$$\frac{\partial u_1(r,\tau)}{\partial \tau} = a\left[\frac{\partial^2 u_1(r,\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial u_1(r,\tau)}{\partial r}\right] - mu_1(r,\tau) + \frac{q(\tau)}{C}, \quad 0 \leqslant r \leqslant R, \tag{1}$$

$$\frac{\partial u_2(r,\tau)}{\partial \tau} = a\left[\frac{\partial^2 u_2(r,\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial u_2(r,\tau)}{\partial r}\right] - mu_2(r,\tau), \quad r \geqslant R, \tag{2}$$

$$m = \frac{2\alpha}{C} = \frac{2\alpha}{c_V\delta}, \tag{3}$$

$$u_1(R,\tau) = u_2(R,\tau), \tag{4}$$

$$\left.\frac{\partial u_1(R,\tau)}{\partial r}\right|_{r=R} = \left.\frac{\partial u_2(R,\tau)}{\partial r}\right|_{r=R}.$$

The effect of the energy source ceases at a considerable distance from it, and so

$$\left.\frac{\partial u_2(r,\tau)}{\partial r}\right|_{r=\infty} = 0. \tag{5}$$

From the condition of axial symmetry,

$$\left.\frac{\partial u_1(r,\tau)}{\partial r}\right|_{r=0} = 0. \tag{6}$$

At time zero the plate temperature is equal to that of the surrounding medium, assumed to be zero; i.e.,

$$u_1(r,0) = u_2(r,0) = t_c = 0. \tag{7}$$

A Laplace integral transformation is used to solve the system (1)–(7).

Applying a Laplace transformation and solving the equations obtained, we may find the following expressions for the transforms $U_1(r, s)$ and $U_2(r, s)$ of the temperatures $u_1(r, \tau)$ and $u_2(r, \tau)$:

$$U_1(r,s) = [1 - \mu K_1(\mu) I_0(\mu\rho)]\frac{W(s)}{s+m}, \quad 0 \leqslant \rho \leqslant 1, \tag{8}$$

$$U_2(r,s) = \mu I_1(\mu) K_0(\mu\rho)\frac{W(s)}{s+m}, \quad \rho \geqslant 1, \tag{9}$$

where

$$\mu = \frac{R}{\sqrt{a}}\sqrt{s+m}, \tag{10}$$

and W(s) is the transform of the function $q(\tau)/C$.

In the special case when we neglect heat losses from the surface to the medium, i.e., we take $a = 0$, Eqs. (8) and (9) may be rewritten in the form

$$U_1(r,s) = [1 - \nu K_1(\nu) I_0(\nu\rho)]\frac{W(s)}{s}, \tag{11}$$

$$U_2(r,s) = \nu I_1(\nu) K_0(\nu\rho)\frac{W(s)}{s}, \tag{12}$$

where $\nu = R/\sqrt{as}$, $\rho = r/R$.

The labor of going from Eqs. (11) and (12) to the functions $u_1(r, \tau)$ and $u_2(r, \tau)$ depends appreciably on the form of the function $q(\tau)$, and we shall therefore restrict ourselves to the simplest form of the impulse, when

$$q(\tau) = \begin{cases} q_0, & 0 \leqslant \tau \leqslant \tau_i, \\ 0, & \tau > \tau_i. \end{cases} \tag{13}$$

Applying the Laplace transformation to the function (13), we obtain its transform

$$Q(s) = \frac{q_0}{s}[1 - \exp(-\tau_i s)]. \tag{14}$$

The transform $U_1(0, s)$ for the temperature $u_1(0, \tau)$ at the center of the plate is obtained from (11) and (14), and has the form

$$U_1(0, s) = \frac{q_0}{C}[1 - \nu K_1(\nu)]\frac{1 - \exp(-\tau_i s)}{s^2}, \quad (15)$$

from which, going to the originals, we find the following expression for determining the temperature at the plate center at time $0 \le \tau \le \tau_i$, when an energy source $q(\tau)$ is acting:

$$u_1(0, \tau) = \frac{q_0\tau}{C}\{[1 - \exp(-z)] - z\,\mathrm{Ei}(-z)\}, \quad (16)$$

where $z = R^2/4a\tau$.

For any time, including also $\tau > \tau_i$, the following expression is valid:

$$u_1(0, \tau) = \frac{q_0\tau}{C}\Big\{1 - \exp(-z) - z\,\mathrm{Ei}(-z) - \sigma_0(\tau - \tau_i)\left(1 - \frac{\tau_i}{\tau}\right)[1 - \exp(-Z) - Z\,\mathrm{Ei}(-Z)]\Big\}, \quad (17)$$

where

$$Z = \frac{1}{(1 - \tau_i/\tau)z},$$

and

$$\sigma_0(\tau - \tau_i) = \begin{cases} 0 & \text{for } 0 \leqslant \tau \leqslant \tau_i, \\ 1 & \text{for } \tau > \tau_i. \end{cases}$$

If the duration of the impulse, t_i, is such that $\exp(-m\tau_i) \approx 1$, then the restriction that the plate is adiabatic may be removed, and the following equation used to calculate the temperature:

$$u_1(0, \tau) = \frac{q_0}{C}\tau\exp(-m\tau)\Big\{1 - \exp(-z) - z\,\mathrm{Ei}(-z) - \sigma_0(\tau - \tau_i)\left(1 - \frac{\tau_i}{\tau}\right)[1 - \exp(-Z) - Z\,\mathrm{Ei}(-Z)]\Big\}. \quad (18)$$

For small values of the parameter $\mathrm{Fo} = a\tau/R^2$, Eqs. (16)–(18) may be transformed to a simpler form, using an asymptotic expansion of the integral exponent function. For example, when $\mathrm{Fo} \le 0.2$ we may replace (16) by the simpler expression

$$u_1(0, \tau) = \frac{q_0\tau}{C}\left[1 - \frac{4a\tau}{R^2}\exp\left(-\frac{R^2}{4a\tau}\right)\right].$$

To evaluate the kinetics of the process, it is sometimes desirable to know the ratio between the amount of heat that has arrived on the plate through its surface, and the heat energy that has been transmitted up to time τ through the boundary of the region of action of the source.

The temperature gradient at the boundary of the regions in the transforms has the form

$$\left.\frac{\partial U_1(r, s)}{\partial r}\right|_{r=R} = \left.\frac{\partial U_2(r, s)}{\partial r}\right|_{r=R} = -\frac{\nu^2}{R}I_1(\nu)K_1(\nu)\frac{W(s)}{s}, \quad (19)$$

from which the total amount of heat, $Q_\lambda(s)$, that has passed in time 0 to τ through the boundary of the region $r = R$, will be

$$Q_\lambda(s) = 2\pi R^2 q_0\frac{I_1(\nu)K_1(\nu)W(s)}{s} \quad (20)$$

To go over to the originals we shall use an asymptotic expansion of the cylindrical functions [1]

$$I_1(\nu)K_1(\nu) \approx \frac{1}{2\nu}\left(1 - \frac{3}{8}\frac{1}{\nu^2}\right). \quad (21)$$

Substituting (21) into (20) and passing to the original, we obtain, finally

$$Q_\lambda(\tau) = Q(\tau)\frac{4}{3\sqrt{\pi}}\sqrt{\mathrm{Fo}}(1 - 0.15\,\mathrm{Fo}). \quad (22)$$

The total amount of heat given out from the zone of action of the energy source through the area $2\pi R\delta$, is determined for any time by the equation

$$Q_\lambda(\tau) = \frac{4}{3}q_0R\tau\sqrt{\pi a\tau}\Big\{1 - 0.15\,\mathrm{Fo} - \sigma_0(\tau - \tau_i)\times\left(1 - \frac{\tau_i}{\tau}\right)^{3/2}\left[1 - 0.15\,\mathrm{Fo}\left(1 - \frac{\tau_i}{\tau}\right)\right]\Big\}. \quad (23)$$

As was the previous one, this equation is valid when $\mathrm{Fo} \le 0.5$. To determine the temperature at the boundary of the source region, we use the approximation

$$\nu I_1(\nu)K_0(\nu) \approx \frac{1}{2}\left(1 - \frac{1}{2}\frac{1}{\nu} - \frac{3}{16}\frac{1}{\nu^3}\right). \quad (24)$$

Substituting (24) into (11) and going from the transforms to the originals, we obtain

$$u_1(R, \tau) = \frac{q_0\tau}{2C}(1 - 0.376\sqrt{\mathrm{Fo}} - 0.0564\,\mathrm{Fo}^{3/2}),$$

$$0 \leqslant \tau \leqslant \tau_i, \quad \mathrm{Fo} < 1. \quad (25)$$

For time values $\tau \lessgtr \tau_i$

$$u_1(R, \tau) = \frac{q_0\tau}{2C}\Big\{1 - 0.376\sqrt{\mathrm{Fo}} - 0.0564\,\mathrm{Fo}^{3/2} - \sigma_0(\tau - \tau_i)\left(1 - \frac{\tau_i}{\tau}\right)\left[1 - 0.376\sqrt{\frac{a(\tau - \tau_i)}{R^2}} - 0.0564\left(\frac{a(\tau - \tau_i)}{R^2}\right)^{3/2}\right]\Big\}. \quad (26)$$

The region of action of the last two equations is limited to values $\mathrm{Fo} = a\tau/R^2 > 1$.

The temperature distribution beyond the source boundary may be found in a manner similar to the foregoing. By replacing the function $\nu I_1(\nu)K_0(\nu\rho)$ by the expression which approximates to it, and going from the transform to the original, for $0 < \tau < \tau_i$, we find

$$u_2(\rho, \mathrm{Fo}) = \frac{2q_0\tau}{C\sqrt{\rho}}(i^2\,\mathrm{erfc}\,x - A_1\sqrt{\mathrm{Fo}}\,i^3\,\mathrm{erfc}\,x - A_2\,\mathrm{Fo}\,i^4\,\mathrm{erfc}\,x - A_3\,\mathrm{Fo}^{3/2}\,i^5\,\mathrm{erfc}\,x), \quad (27)$$

where

$$x = \frac{\rho - 1}{2\sqrt{\mathrm{Fo}}} = \frac{r - R}{2\sqrt{a\tau}},$$

$$A_1 = \frac{1}{4}\left(3 + \frac{1}{\rho}\right), \quad A_2 = \frac{1}{32}\left(15 - \frac{6}{\rho} - \frac{9}{\rho^2}\right),$$

$$A_3 = \frac{1}{128}\left(105 - \frac{15}{\rho} + \frac{27}{\rho^2} + \frac{75}{\rho^3}\right).$$

The functions $i^n\mathrm{erfc}\, x$ have been examined and tabulated in the appendix of the monograph of reference [1]. To calculate the heat transfer with the surrounding medium, the right side of (27) must be multiplied by $\exp(-m\tau)$ (if $\exp(-m\tau_i) \approx 1$ and Fo < 1).

NOTATION

u_1 and u_2 are the temperature of the first and second zones of the plate; r is a coordinate; τ is the time; R is the radius of the source; λ and a are the thermal conductivity and diffusivity; c_V is the volume specific heat; δ is the plate thickness; $C = c_V\delta$ is the heat capacity per unit area of the plate; α is the heat transfer coefficient; $m = 2\alpha/C$ is the heat transfer rate; $\varepsilon = 1/m$ is the coefficient of thermal inertia of the plate; q is the heat flux; $\rho = r/R$ is the dimensionless coordinate; s is the Laplace transform parameter; $I_0(z)$, $I_1(z)$, $K_0(z)$, $K_1(z)$, are the modified Bessel functions of the first and second kind and of order zero and one; $\mathrm{Ei}(-z)$ is the integral indicial function of argument $z > 0$; $Q(\tau) = q_0 2\pi R^2 \tau$ is the amount of heat energy liberated in the source zone in time τ; $Q_\lambda(\tau)$ is the amount of energy passing across the source boundary.

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